The Hopf Fibration: An Intuitive Understanding

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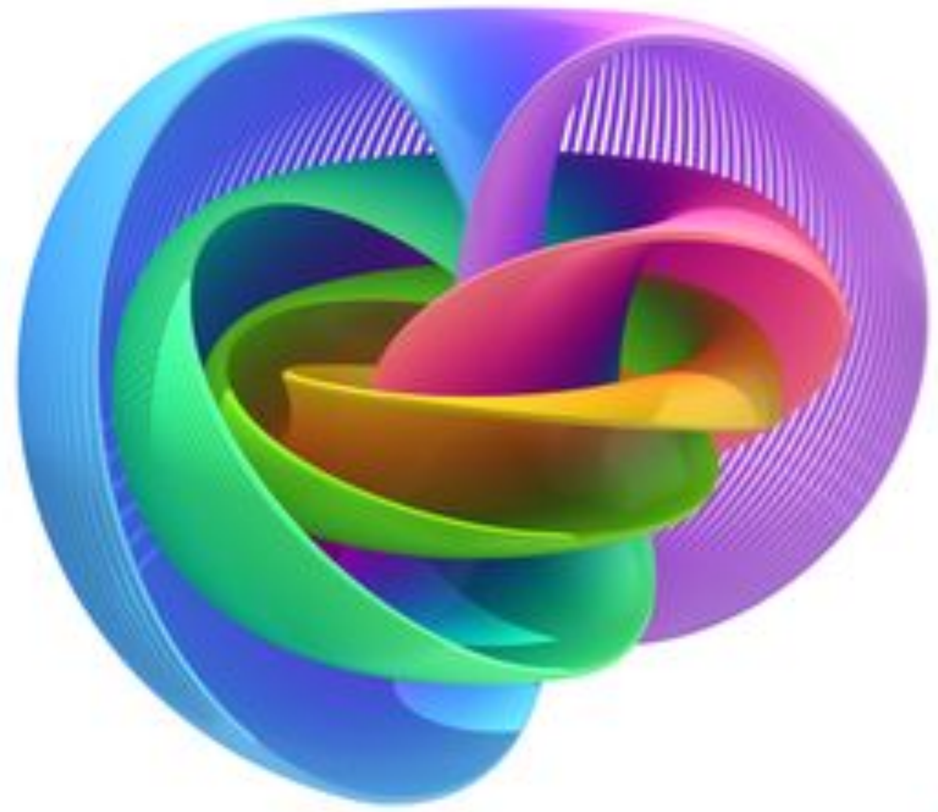


Abstract

This poster will show how the Hopf Fibration can be understood intuitively with out the need for a background in topology. The hopf fibration is a mapping from 4 dimensional space into 3 dimensional space which is difficult to visualize making it a hard concept to grasp. However through techniques such as stereographic projection and some insight into the 3-sphere this poster will provide intuition to an idea deeply intertwined with quantum physics and the calculations of rotations in 3 dimensional space.

Introduction to Spheres

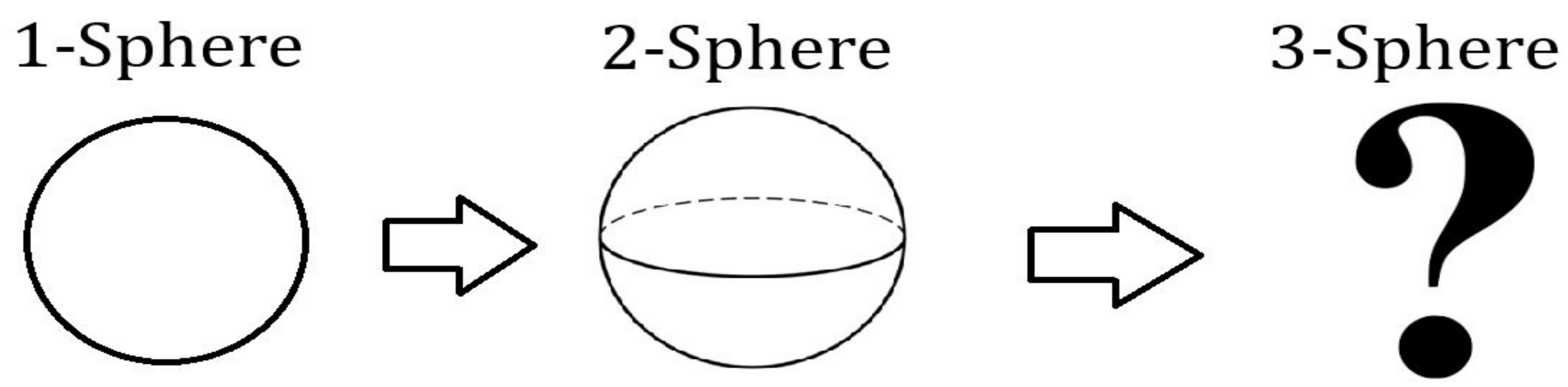
The Hopf Fibration describes a mapping from the 3-sphere to the 2-sphere, but what do these mean? The 2-sphere: S^2 is given by the equation

$$S^2 = \{x_0, x_1, x_2 \in \mathbb{R}^3 | \sqrt{x_0^2 + x_1^2 + x_2^2} = 1\}$$

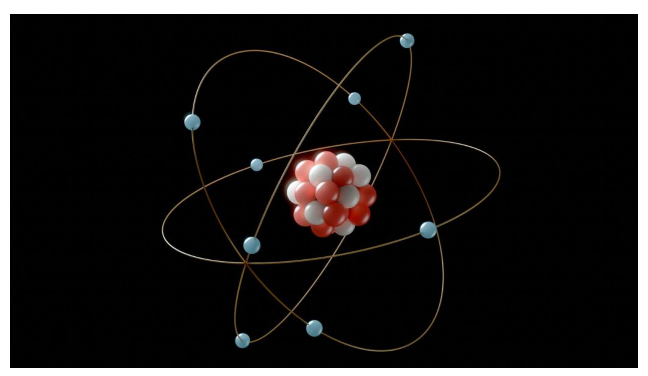
The 3-sphere: S^3 is given by the equation

$$S^3 = \{x_0, x_1, x_2, x_3 \in \mathbb{R}^4 | \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2} = 1\}$$

Notice how the 2-sphere lies in 3 dimensions and the 3-sphere lies in 4 dimensions.



Real World Application



In computer graphics and robotics, quaternions—closely tied to the Hopf fibration—are widely used to represent and compute 3D rotations. Quantum physics is also heavily affected by The Hopf fibration allows us to bridge these electrons from the quantum realm where their phase shift matters our physical world where they don't.

Stereographic Projection

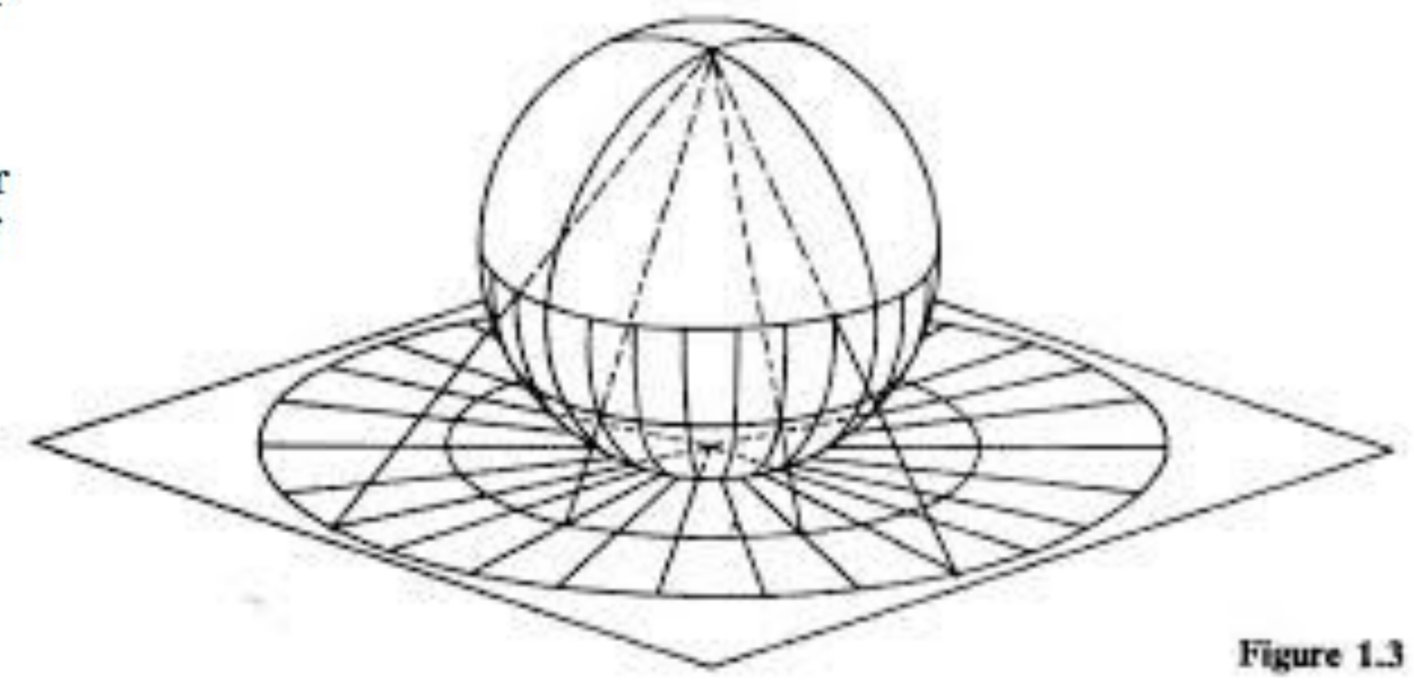


Figure 1.3

Stereographic projection is the process of taking a 3 dimensional sphere and "flattening it out" to a 2 dimensional plane. The equation for this is

$$(x, y, z) \to (\frac{x}{1-z}, \frac{y}{1-z})$$

Since the 3-sphere lies in 4 dimensional space we need to use stereographic projection to "flatten it out" into a 3 dimensional space. The equation for this is

$$(w, x, y, z) \to (\frac{x}{1-w}, \frac{y}{1-w}, \frac{z}{1-w})$$

The picture you see is just a projection of what the 4 dimensional sphere looks like in 3 dimensions

Hopf Mapping with Real Numbers

The Hopf Mapping described with real numbers is defined by

$$h(a, b, c, d) = (a^2 + b^2 - c^2 - d^2, 2(ad + bc), 2(bd - ac))$$

This mapping shows how the points get mapped but it doesn't really provide much intuition into why it works

Hopf Mapping with Complex Numbers

The Hopf Mapping described with complex numbers is defined by

$$h(z_1, z_2) = (2z_1\overline{z_2}, |z_1|^2 - |z_2|^2)$$

Now we've simplified the mapping considerably but we still don't have any insight as to what the mapping is doing.

Acknowledgments:

A Young Person's Guide to the Hopf Fibration by Zachary Treisman
An Elementary Introduction to the Hopf Fibration by David W. Lyons
Niles Johnson: Visualisations of the Hopf Fibration
A Quick Intro to Fiber Bundles (Hopf Fibration) by Richard Behiel
Notes On The Course "Algebraic Topology", 2019-2020 by Boris Botvinnik