Stereographic Projection

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Introduction

A stereographic projection is a projection of the unit sphere onto a plane through the north pole via a bijective mapping from the sphere to the set of real or complex numbers extended by an infinity point.

Infinity Point

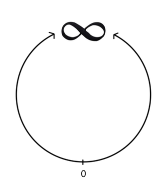

Figure 1. $\mathbb{R} \cup \{\infty\}$

- The infinity point is necessary for the bijective mapping. Without the infinity point, the stereographic projection would fail to provide a one-to-one correspondence between the sphere and the plane.
- The infinity point is the image of the north pole of the sphere. As the mapping uses similar triangles for every non-north pole point, there is no map from the north pole to the real or complex numbers.

Circle

Consider the unit circle $S^1 = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$

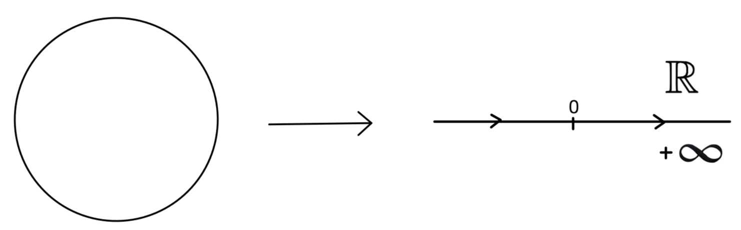

Figure 2. The map $s : S^1 \to \mathbb{R} \cup \{\infty\}$

$$s(x, y) = \begin{cases} \frac{x}{1-y} & y \neq 1 \\ \infty & y = 1 \end{cases}$$

Let $N = (0, 1)$ represent the north pole of the circle.

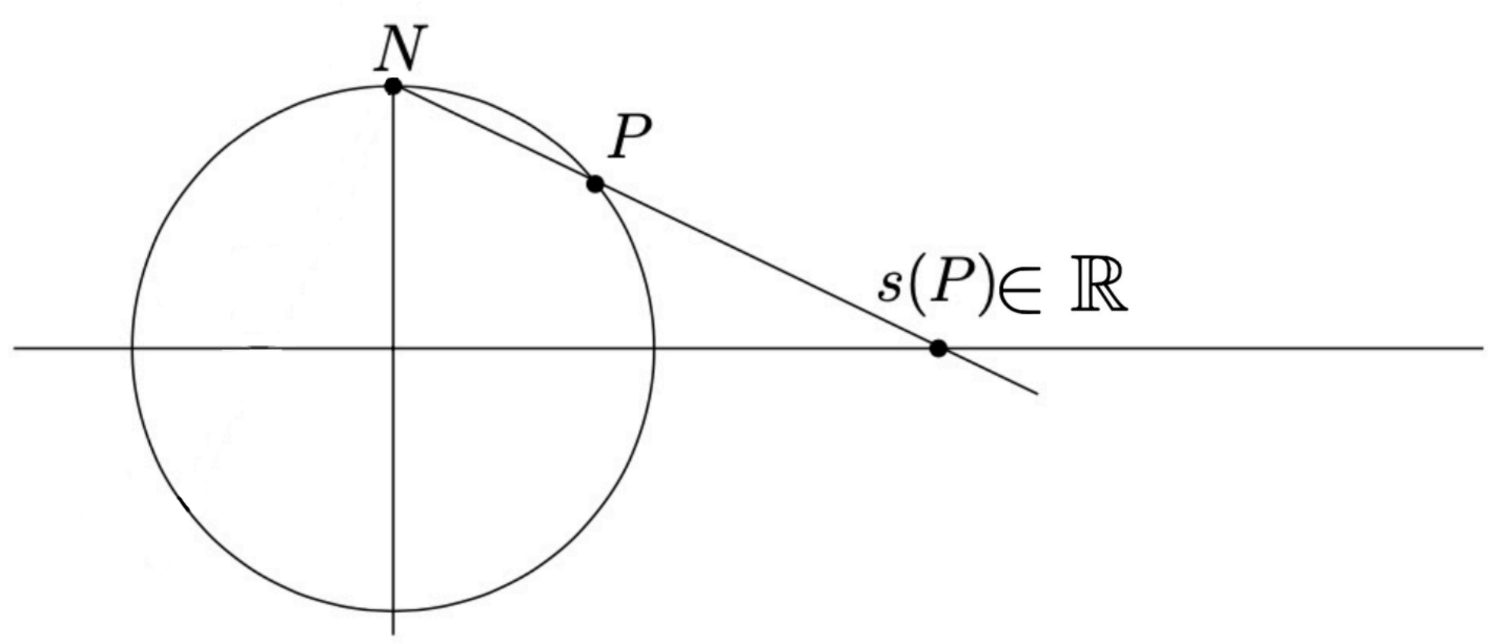

Figure 3. Stereographic Projection of S^1

Given any point P on the circle where $N \neq P$, we can find the line $\overline{NP}$ that goes through the north pole N and the point P. The intersection of the line $\overline{NP}$ and the x-axis, denoted $s(P)$, is the image of the stereographic projection map.

For $r \in \mathbb{R} \cup \{\infty\}$, we can define the inverse map $s^{-1} : \mathbb{R} \cup \{\infty\} \to S^1$ as

$$s^{-1}(r) = \begin{cases} \left(\frac{2r}{r^2+1}, \frac{r^2-1}{r^2+1}\right) & r \neq \infty \\ (0, 1) & r = \infty \end{cases}$$

Sphere

Consider the unit sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}$

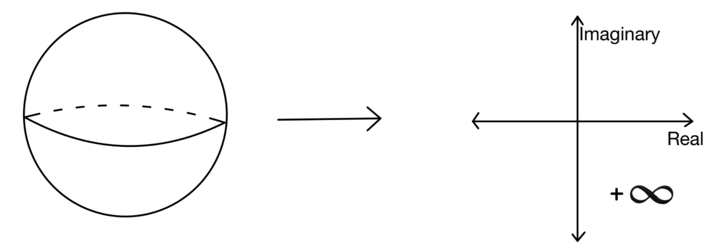

Figure 4. The map $s : S^2 \to \mathbb{C} \cup \{\infty\}$

$$s(x, y, z) = \begin{cases} \frac{x+iy}{1-z} & z \neq 1 \\ \infty & z = 1 \end{cases}$$

Let $N = (0, 0, 1)$ represent the north pole of the sphere.

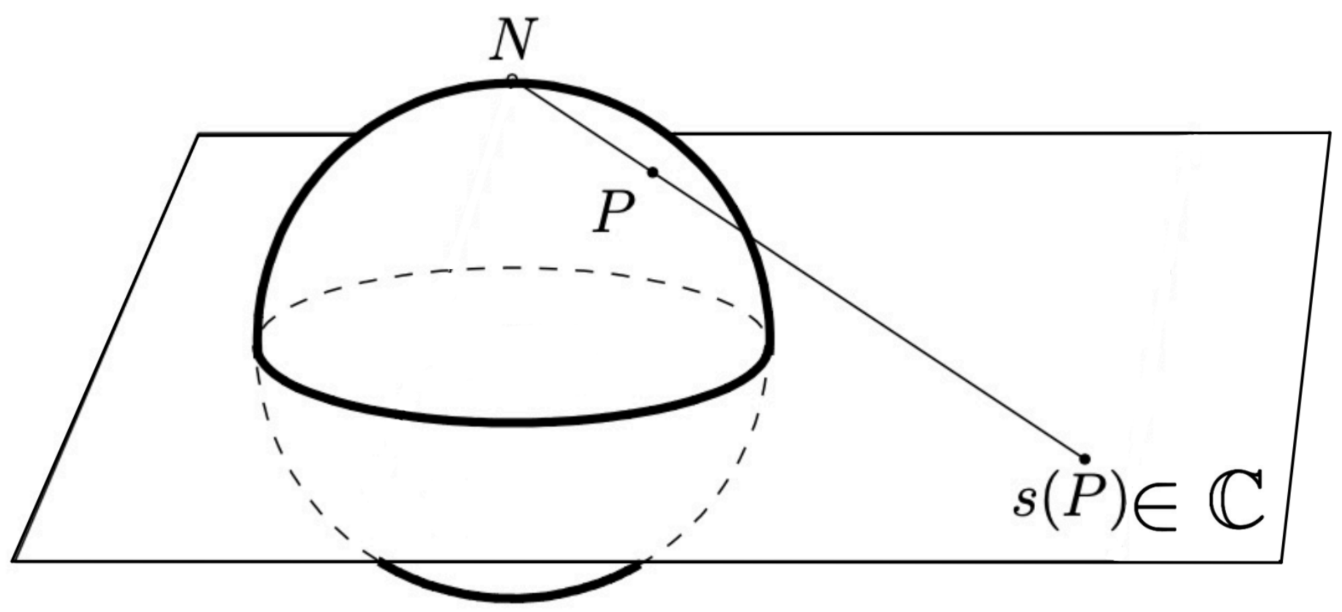

Figure 5. Stereographic Projection of S^2

Given any point P on the sphere where $N \neq P$, we can find the line $\overline{NP}$ that goes through the north pole N and the point P. The intersection of the line $\overline{NP}$ and the complex-plane, denoted $s(P)$, is the image of the stereographic projection map.

For $c \in \mathbb{C} \cup \{\infty\}$, we can define the inverse map $s^{-1} : \mathbb{C} \cup \{\infty\} \to S^2$ as

$$s^{-1}(c) = \begin{cases} \left(\frac{2\text{Re}(c)}{|c|^2+1}, \frac{2\text{Im}(c)}{|c|^2+1}, \frac{|c|^2-1}{|c|^2+1}\right) & c \neq \infty \\ (0, 0, 1) & c = \infty \end{cases}$$

[3]

The Reimann Sphere is the a model of the set of complex numbers extended by an infinity point, $\mathbb{C} \cup \{\infty\}$, visualized by a stereographic projection from the plane to the sphere.

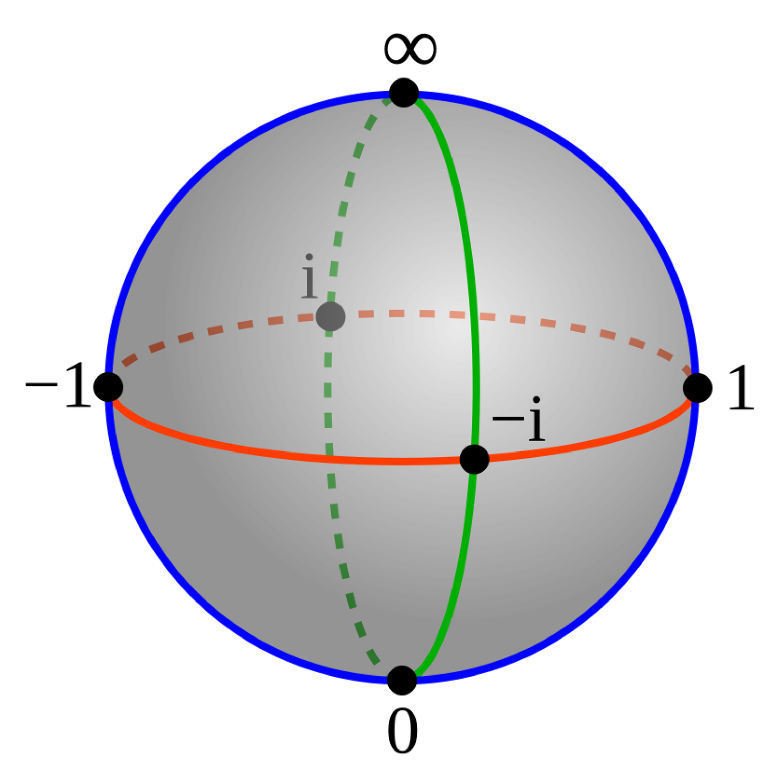

Figure 6. Reimann Sphere [1]

Similar Triangles

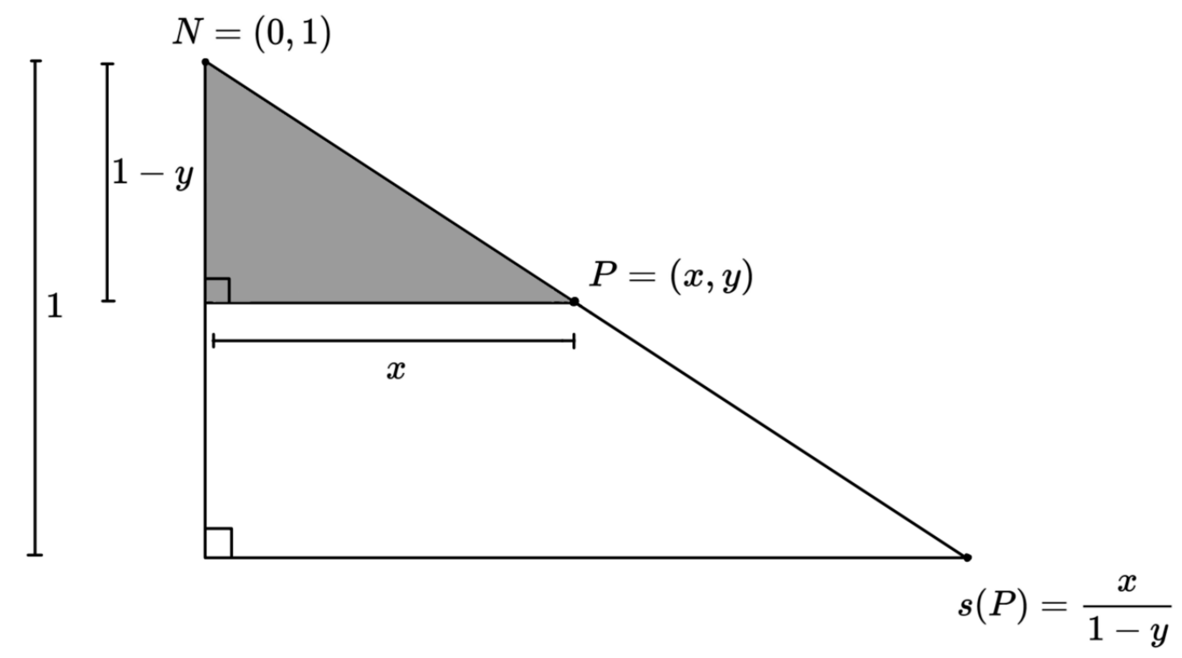

Figure 7. Similar triangles formed by map s

[5]

Properties

- The mapping is conformal: it preserves angles between lines. For a circle $C \in S^2$ that does not intersect the north pole, the image $s(C) \in \mathbb{C} \cup \{\infty\}$ is a circle. If C does intersect the north pole, then $s(C)$ is a line.
- The mapping does not preserve distances or areas. The elements close to the north pole of the circle or sphere will be mapped the furthest from 0 or (0,0) on the real numbers or complex plane.
- For any $\frac{m}{n} \in \mathbb{Q}$ where $\gcd(m, n) = 1$ and $0 < n < m$, the inverse mapping $s^{-1} : \mathbb{R} \cup \{\infty\} \to S^1$, will map $\frac{m}{n}$ to $\left(\frac{2mn}{m^2+n^2}, \frac{m^2-n^2}{m^2+n^2}\right)$, a rational point on S^1 corresponding to the Pythagorean Triple generated by m and n [4].

Applications

- The Stereographic Projection can be used as a tool, similar to a change of basis transformation, as S^n and $\mathbb{R}^n \cup \{\infty\}$ are homeomorphic topological spaces. This allows for the use of the Cartesian Coordinate System.
- Used as a tool in Cartography as it allows for angle-preserving visuals of the globe.
- Used in Crystallography for visualizing and interpreting atomic structures.

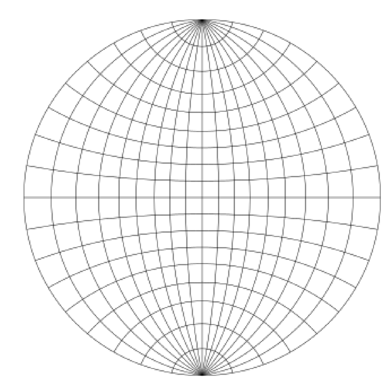

Figure 8. Wuff Net, used for graphing coordinates on the surface of a sphere [2]

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